ON A VARIATIONAL PRINCIPLE OF THE COMPLEX THEORY OF SHELLS

(O VARIATSIONNOM PRINTSIPE KOMPLEKSNOI TEORII OBOLOCHEK)

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When analyzing shells of complicated shape (reservoirs, boilers, etc.) it often becomes necessary to break them up into several simpler parts.

In such cases the use of the complex representation [1] meets with difficulties arising out of the necessity to separate the real and imaginary parts when the solutions are coupled together.

In some special cases [1] complex combinations of the unknown quantities were proposed so that it was possible to avoid these difficulties.

In the present paper the complex coupling conditions of the general form are derived as the natural boundary conditions of the proposed variational problem.

The notion of a complex energy is introduced, and the minimum property of its modulus is demonstrated.

The middle surface of the shell will be referred to the lines of principal curvature (a_1, a_2), and for the sake of brevity it will be assumed that the region of the change of the coordinates of the middle surface S is bounded by some arbitrary closed line of curvature, $a_1 = a_1^0$ = const. for example. It will be also assumed that Poisson's ratio μ is zero.

Then, the generalized Hooke's law relates the forces and moments to the displacements of the middle surface u, v, w in the well-known fashion [1]:

$$\begin{aligned} T_1 &= Eh\varepsilon_1(u, v, w), & M_1 &= Ehc_0^2\varkappa_1(u,v,w), \\ T_2 &= Eh\varepsilon_2(u, v, w) & M_2 &= Ehc_0^2\varkappa_2(u, v, w) \\ S &= Eh\frac{1}{2}\omega(u, v, w), & H &= Ehc_0^2\tau(u, v, w) \end{aligned} \tag{1}$$

Generalizing the notion of the Lur'e-Goldenveizer functions [2, 3] to the case of the non-homogeneous problem of shell theory, functions of stress, a, b, c, will be introduced by the following relations:

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$$
\begin{aligned}
T_1 &= T_1^* + Ehc_0\varkappa_2(a, b, c), & M_1 &= M_1^* - Ehc_0\varepsilon_2(a, b, c)\\
T_2 &= T_2^* + Ehc_0\varkappa_1(a, b, c), & M_2 &= M_2^* - Ehc_0\varepsilon_1(a, b, c)\\
S &= S^* - Ehc_0\tau(a, b, c), & H &= H^* + Ehc_0\frac{1}{2}\omega(a, b, c)
\end{aligned} \tag{2}
$$

Here h = const is the shell thickness, $c_0^2 = 1/12\ h^2$, while the deformations are expressed in terms of the displacements of the middle surface by the following relations:

$$
\begin{aligned}
\varepsilon_1 &= \frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}v + \frac{w}{R_1}, & \varkappa_1 &= \frac{1}{A_1}\frac{\partial \vartheta}{\partial \alpha} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\psi\\
\varepsilon_2 &= \frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}u + \frac{w}{R_2}, & \varkappa_2 &= \frac{1}{A_2}\frac{\partial \psi}{\partial \alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\vartheta
\end{aligned}
$$

$$
\omega = \omega_1 + \omega_2, \quad \tau = \tau_2 + \frac{\omega_1}{R_2} = \tau_1 + \frac{\omega_2}{R_1} \tag{3}
$$

where

$$
\begin{aligned}
\vartheta &= -\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} + \frac{u}{R_1}, & \omega_1 &= \frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}u\\
\psi &= -\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} + \frac{v}{R_2}, & \omega_2 &= \frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}v
\end{aligned}
$$

are the angles of rotation of the respective tangents to the coordinate lines α_1 and α_2 around the directions shown in Fig. 1.,

$$
\tau_1 = \frac{1}{A_1}\frac{\partial \psi}{\partial \alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\vartheta, \qquad \tau_2 = \frac{1}{A_2}\frac{\partial \vartheta}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\psi
$$

and A_1, A_2, R_1, R_2 are Lame's constants and the principal radii of curvature of the middle surface, respectively. A certain solution of the equilibrium equations (a statically allowable set of functions) is chosen as a set of functions (T_1^*, T_2^*, S^*, M_1^*, M_2^*, H^*). The following complex combinations are called complex forces:

$$
\begin{aligned}
\check{T}_1 &= T_1 - iEhc_0\varkappa_2 = T_1 - \frac{i}{c_0}M_2\\
\check{T}_2 &= T_2 - iEhc_0\varkappa_1 = T_2 - \frac{i}{c_0}M_1\\
\check{S} &= S + iEhc_0\tau = S + \frac{i}{c_0}H
\end{aligned} \tag{4}
$$

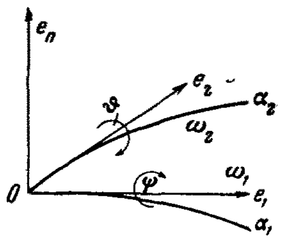

Fig. 1.

They satisfy the following system of equations on complex forces [1]:

$$
\frac{\partial A_2\check{T}_1}{\partial \alpha_1} + \frac{1}{A_1}\frac{\partial A_1^2\check{S}}{\partial \alpha_2} - \frac{\partial A_2}{\partial \alpha_1}\check{T}_2 + \tag{5}
$$

$$
+ \frac{ic_0}{R_1}\left[\frac{\partial A_2\check{T}_2}{\partial \alpha_1} - \frac{1}{A_1}\frac{\partial A_1^2\check{S}}{\partial \alpha_2} - \frac{\partial A_2}{\partial \alpha_1}\check{T}_1\right] + \prod\nolimits_1 + A_1A_2q_1 = 0
$$

$$\frac{\partial A_1 T_2^{\vee}}{\partial \alpha_2} + \frac{1}{A_2}\frac{\partial A_2^2 S^{\vee}}{\partial \alpha_1} - \frac{\partial A_1}{\partial \alpha_2} T_1^{\vee} + \\ + \frac{ic_0}{R_2}\left[\frac{\partial A_1 T_1^{\vee}}{\partial \alpha_2} - \frac{1}{A_2}\frac{\partial A_2^2 S^{\vee}}{\partial \alpha_1} - \frac{\partial A_1}{\partial \alpha_2} T_2^{\vee}\right] + \prod_2 + A_1 A_2 q_2 = 0$$

$$\frac{T_1^{\vee}}{R_1} + \frac{T_2^{\vee}}{R_2} - \frac{ic_0}{A_1 A_2}\left\{\frac{\partial}{\partial \alpha_1}\frac{1}{A_1}\left(\frac{\partial A_2 T_2^{\vee}}{\partial \alpha_1} - \frac{1}{A_1}\frac{\partial A_1^2 S^{\vee}}{\partial \alpha_2} - \frac{\partial A_2}{\partial \alpha_1} T_1^{\vee}\right) + \\ + \frac{\partial}{\partial \alpha_2}\frac{1}{A_2}\left(\frac{\partial A_1 T_1^{\vee}}{\partial \alpha_2} - \frac{1}{A_2}\frac{\partial A_2^2 S^{\vee}}{\partial \alpha_1} - \frac{\partial A_1}{\partial \alpha_2} T_2^{\vee}\right)\right\} - q_n = 0$$

where

$$\prod_1 = ic_0\left\{-\frac{1}{R_2}\frac{\partial A_1}{\partial \alpha_2} S^{\vee} - \frac{\partial}{\partial \alpha_2}\left(\frac{A_1}{R_1} S^{\vee}\right)\right\}, \qquad \prod_2 = ic_0\left\{-\frac{1}{R_1}\frac{\partial A_2}{\partial \alpha_1} S^{\vee} - \frac{\partial}{\partial \alpha_1}\left(\frac{A_2}{R_2} S^{\vee}\right)\right\}$$

This system is equivalent to the aggregate of the sets of the equilibrium and continuity equations of the middle surface. This can be easily shown by separating the real and imaginary parts.

Complex displacements are introduced by the relations:

$$u^{\vee} = u + ia, \qquad v^{\vee} = v + ib, \qquad w^{\vee} = w + ie \tag{6}$$

When the complex combinations of the corresponding equations of systems (1) and (2) are constructed, a relation between the complex forms and displacements is obtained:

$$\begin{aligned} T_1^{\vee} &= T_1^* - iEhc_0\varkappa_2^{\vee}(u^{\vee}, v^{\vee}, w^{\vee}), & T_1^{\vee} &= -\frac{i}{c_0} M_2^* + Eh\varepsilon_1^{\vee}(u^{\vee}, v^{\vee}, w^{\vee}) \\ T_2^{\vee} &= T_2^* - iEhc_0\varkappa_1^{\vee}(u^{\vee}, v^{\vee}, w^{\vee}), & T_2^{\vee} &= -\frac{i}{c_0} M_1^* + Eh\varepsilon_2^{\vee}(u^{\vee}, v^{\vee}, w^{\vee}) \\ S^{\vee} &= S^* + iEhc_0\tau^{\vee}(u^{\vee}, v^{\vee}, w^{\vee}), & S^{\vee} &= \frac{i}{c_0} H^* + Eh\frac{1}{2}\omega^{\vee}(u^{\vee}, v^{\vee}, w^{\vee}) \end{aligned} \tag{7}$$

Finally, eliminating from here the complex forces $T_1^{\vee}$, $T_2^{\vee}$, $S^{\vee}$, a set of equations on the complex displacements is obtained:

$$\begin{aligned} \varepsilon_1^{\vee} + ic_0\varkappa_2^{\vee} &= \frac{1}{Eh}\left(T_1^* + \frac{i}{c_0} M_2^*\right) \\ \varepsilon_2^{\vee} + ic_0\varkappa_1^{\vee} &= \frac{1}{Eh}\left(T_2^* + \frac{i}{c_0} M_1^*\right) \\ \frac{\omega^{\vee}}{2} - ic_0\tau^{\vee} &= \frac{1}{Eh}\left(S^* - \frac{i}{c_0} H^*\right) \end{aligned} \tag{8}$$

If the general solution of the membrane theory (T_1^*, T_2^*, S^*) is chosen as the set of functions $(T_1^*, T_2^*, S^*, M_1^*, M_2^*, H^*)$ then sets (7) and (8) will coincide with analogous sets proposed by Novozhilov.

Four complex quantities are to be associated with each point of the boundary $(\alpha_1 = \alpha_1^0)$:

$$T_1^{\vee}, \quad T_{12}^{\vee\prime} = S^{\vee} - ic_0 \frac{2S^{\vee}}{R_2}, \quad N_1^{\vee\prime} = \frac{ic_0}{A_1 A_2}\left(\frac{\partial A_2 T_2^{\vee}}{\partial \alpha_1} - 2\frac{\partial A_1 S^{\vee}}{\partial \alpha_2} - \frac{\partial A_2}{\partial \alpha_1} T_1^{\vee}\right)$$

$$M_1^{\vee} = ic_0 T_2^{\vee} \tag{9}$$

Separating out in the reduced complex expressions the real parts, the following values are obtained:

$$T_1, \qquad T_{12}' = S + 2\frac{H}{R_2} = T_{12} + \frac{M_{12}}{R_2}$$

$$N_1' = \frac{1}{A_1 A_2}\left(\frac{\partial A_2 M_1}{\partial \alpha_1} + 2\frac{\partial A_1 H}{\partial \alpha_2} - \frac{\partial A_2}{\partial \alpha_1} M_2\right) = N_1 + \frac{1}{A_1}\frac{\partial M_{12}}{\partial \alpha_2}, \qquad M_1$$

These, however, are the boundary forces generalized according to Kirchhoff (see [1], p. 55).

The imaginary parts of the expressions in (9) are:

$$-Ehc_0 \varkappa_2, \qquad Ehc_0 \varkappa_{21}' = Ehc_0\left\{\tau - \frac{\omega}{R_2}\right\}$$

$$-Ehc_0 \zeta_2' = -Ehc_0\left\{-\frac{1}{A_1 A_2}\left(\frac{\partial A_2 \varepsilon_2}{\partial \alpha_1} - \frac{1}{A_1}\frac{\partial A_1 \omega}{\partial \alpha_2} - \frac{\partial A_2}{\partial \alpha_1}\varepsilon_1\right)\right\}, \qquad Ehc_0 \varepsilon_2$$

In order to explain the geometric meaning of the quantities κ_2, κ_{21}', ζ_2', ϵ_2 they shall be changed somewhat. Substituting into the third relation the expressions for the deformations (3) one obtains with the aid of the Codazzi-Gauss relations

$$\zeta_2' = \frac{\vartheta}{R_2} + \frac{1}{A_2}\frac{\partial \omega_2}{\partial \alpha_2}$$

Furthermore, $\tau = \tau_2 + \omega_1/R_2$, $\omega = \omega_1 + \omega_2$; hence

$$\varkappa_{21}' = \tau_2 - \frac{\omega_2}{R_2}$$

Thus there are four quantities fixed at the neutral line:

$$\varepsilon_2, \varkappa_2, \zeta_2' = \frac{\vartheta}{R_2} + \frac{1}{A_2}\frac{\partial \omega_2}{\partial \alpha_2}, \qquad \varkappa_{21}' = \tau_2 - \frac{\omega_2}{R_2} \tag{10}$$

It will be shown that the fixing of these quantities completely determines the deformation of an element of the boundary. Isolate for that purpose an element of the boundary, connected with an element of the arc.

First of all it is clear that ϵ_2 gives the relative elengation of the neutral line, and κ_2 gives its curvature in the plane of the diagram.

The curvature in the direction perpendicular to the plane of the diagram will be found by determining it as the limit of the ratio of the difference of the angles at two adjacent points, measured from the same direction (e_n), to the distance between them, i.e. by equating it to:

$$\varkappa_{2z} = \lim_{\Delta\alpha_2 \to 0} \frac{\omega_2'' - \omega_2}{A_2 \Delta\alpha_2} \tag{11}$$

From Fig. 2 it can be seen that

$$\omega_2'' = \left(\omega_2 + \frac{\Delta\omega_2}{\Delta\alpha_2}\Delta\alpha_2\right)\cos\Delta\beta + \left(\vartheta + \frac{\Delta\vartheta}{\Delta\alpha_2}\Delta\alpha_2\right)\sin\Delta\beta$$

Because of the smallness of the angle $\Delta\beta$

$$\cos\Delta\beta \approx 1, \qquad \sin\Delta\beta \approx \Delta\beta \approx \frac{A_2\Delta\alpha_2}{R_2}$$

and

$$\omega_2'' \approx \omega_2 + \left(\frac{\Delta\omega_2}{\Delta\alpha_2} + \frac{A_2}{R_2}\vartheta\right)\Delta\alpha_2$$

Substituting the resulting relation into equation (11) one obtains

$$\varkappa_{2z} = \frac{\vartheta}{R_2} + \frac{1}{A_2}\frac{\partial\omega_2}{\partial\alpha_2} = \zeta_2'$$

Thus the first three quantities determine deformations of an element of the neutral line. Furthermore, [1],

$$\omega_2^{(z)} = \frac{\omega_2 + z\tau_2}{1 + z/R_2} \approx \omega_2 + z\left(\tau_2 - \frac{\omega_2}{R_2}\right)$$

From this it is evident that the fourth quantity characterizes the change of magnitude of the deflection angle ω_2, i.e. the twist of the boundary element.

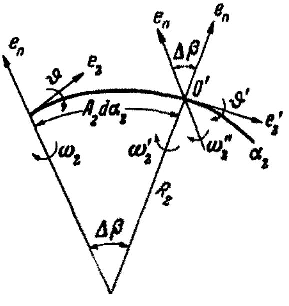

Fig. 2.

According to Kirchhoff's kinematical hypothesis, however, the deformation of the boundary element can be determined by just four quantities. Thus, the fixing of the imaginary parts of the complex expressions (9) completely determines the deformation of the boundary element.

Assuming, as above, that the middle surface of the shell is bounded by a closed line of curvature $\alpha_1 = \alpha_1^0 = \text{const.}$ it will be shown that the following variational equations can be postulated as the basis of the theory of shells:

$$\iint\limits_S \{T_1^{\vee}\delta\varepsilon_1^{\vee}+T_2^{\vee}\delta\varepsilon_2^{\vee}+S^{\vee}\delta\omega^{\vee}+ic_0T_2^{\vee}\delta\varkappa_1^{\vee}+ic_0T_1^{\vee}\delta\varkappa_2^{\vee}-ic_0S^{\vee}\delta\tau^{\vee}\}\,A_1A_2d\alpha_1d\alpha_2\,-$$

$$-\int\limits_{\alpha_1=\alpha_1^0}\{T_1^{\vee}\delta u^{\vee}+T_{12}^{\vee\prime}\delta v^{\vee}+N_1^{\vee\prime}\delta w^{\vee}+M_1^{\vee}\delta\vartheta^{\vee}\}\,A_2d\alpha_2\,-$$

$$-\iint\limits_S\{q_1\delta u^{\vee}+q_2\delta v^{\vee}+q_n\delta w^{\vee}\}\,A_1A_2d\alpha_1d\alpha_2=0 \tag{12}$$

In fact, integrating by parts the first integral, it can be easily shown that the system of equations on complex forces (5), which is equivalent to the combination of the sets of equations of equilibrium and continuity of the middle surface, will appear as the Euler equations for (12).

Using relations (7), equation (12) can be changed to

$$\delta V^{\vee}-\int\limits_{\alpha_1=\alpha_1^0}\{T_1^{\vee}\delta u^{\vee}+T_{12}^{\vee\prime}\delta v^{\vee}+N_1^{\vee\prime}\delta w^{\vee}+M_1^{\vee}\delta\vartheta^{\vee}\Big\}\,A_2d\alpha_2\,-$$

$$-\,\delta\iint\limits_S\{q_1u^{\vee}+q_2v^{\vee}+q_nw^{\vee}\}\,A_1A_2d\alpha_1d\alpha_2=0$$

Here the quantity

$$V^{\vee}=\frac{Eh}{2}\Big\{\iint\limits_S\Big\{(\varepsilon_1^{\vee}+\varepsilon_2^{\vee})^2-2\Big[\varepsilon_1^{\vee}\varepsilon_2^{\vee}-\Big(\frac{\omega^{\vee}}{2}\Big)^2\Big]\Big\}\,A_1A_2d\alpha_1d\alpha_2\,+$$

$$+\frac{h^2}{12}\iint\limits_S\Big\{(\varkappa_1^{\vee}+\varkappa_2^{\vee})^2-2\,[\varkappa_1^{\vee}\varkappa_2^{\vee}-\tau^{\vee 2}]\Big\}\,A_1A_2d\alpha_1d\alpha_2\Big\}\Big\}+$$

$$+\iint\limits_S\Big\{-\frac{i}{c_0}M_2^*\varepsilon_1^{\vee}-\frac{i}{c_0}M_1^*\varepsilon_2^{\vee}+\frac{i}{c_0}H^*\omega^{\vee}+ic_0T_2^*\varkappa_1^{\vee}+ic_0T_1^*\varkappa_2^{\vee}-ic_02S^*\tau\Big\}\,A_1A_2d\alpha_1d\alpha_2-$$

$$-\frac{1}{2Eh}\Big\{\iint\limits_S\Big\{(T_1^*+T_2^*)^2-2\,[T_1^*T_2^*-S^{*2}]\Big\}\,A_1A_2d\alpha_1d\alpha_2\,+$$

$$+\frac{12}{h^2}\iint\limits_S\Big\{(M_1^*+M_2^*)^2-2\,[M_1^*M_2^*-H^{*2}]\Big\}A_1A_2d\alpha_1d\alpha_2\Big\}\Big\}$$

can be called by analogy the potential energy of the complex deformation, (note that the quantities ($T_1{}^*$, $T_2{}^*$, ..., H^*) are not being varied).

Of course, there are also other possible forms of complex variational equations, reducing to one or another set of natural boundary conditions. It should be noted that a peculiar minimum property of the potential energy of the complex deformation is in effect.

Namely, the following takes place:

For the actual complex displacements, i.e. for u, v, w, which satisfy the system of equations on complex displacements (8), the potential energy of the complex deformation V approaches a minimum value (in modulus).

Actually, V can be written as follows:

$$V^{\vee} = \frac{Eh}{2}\iint_S \Big\{\Big[\varepsilon_1^{\vee} + ic_0\varkappa_2^{\vee} - \frac{1}{Eh}\Big(T_1^{*} + \frac{i}{c_0}M_2^{*}\Big)\Big]\Big[\varepsilon_1^{\vee} - ic_0\varkappa_2^{\vee} + \frac{1}{Eh}\Big(T_1^{*} - \frac{i}{c_0}M_2^{*}\Big)\Big] +$$

$$+\Big[\varepsilon_2^{\vee} + ic_0\varkappa_1^{\vee} - \frac{1}{Eh}\Big(T_2^{*} + \frac{i}{c_0}M_1^{*}\Big)\Big]\Big[\varepsilon_2^{\vee} - ic_0\varkappa_1^{\vee} + \frac{1}{Eh}\Big(T_2^{*} - \frac{i}{c_0}M_1^{*}\Big)\Big] +$$

$$+2\Big[\frac{\omega^{\vee}}{2} - ic_0\tau^{\vee} - \frac{1}{Eh}\Big(S^{*} - \frac{i}{c_0}H^{*}\Big)\Big]\Big[\frac{\omega^{\vee}}{2} + ic_0\tau^{\vee} + \frac{1}{Eh}\Big(S^{*} + \frac{i}{c_0}H^{*}\Big)\Big]\Big\} A_1A_2d\alpha_1d\alpha_2$$

From this it can be seen that for $u^{\vee}$, $v^{\vee}$, $w^{\vee}$, which satisfy the set of equations (8), $V^{\vee} = 0$. This then proves the minimum property.

As a first application, the variational equation will be used to obtain the complex coupling conditions along a line of curvature of two shells of the same thickness.

Let two shells be coupled along line l. Assign the plus sign to all quantities relating to the first shell, and the minus sign to all quantities relating to the second shell, and take the following variational equation:

$$\iint_{S^+}\{T_1^{\vee +}\delta\varepsilon_1^{\vee +} + T_2^{\vee +}\delta\varepsilon_2^{\vee +} + S^{+}\delta\omega^{+\vee} + ic_0T_2^{\vee +}\delta\varkappa_1^{\vee +} + ic_0T_1^{\vee +}\delta\varkappa_2^{\vee +} - ic_0S^{\vee +}\delta\tau^{+\vee}\}A_1^{+}A_2^{+}d\alpha_1^{+}d\alpha_2^{+} -$$

$$-\iint_{S^+}\{q_1^{+}\delta u^{+\vee} + q_2^{+}\delta v^{+\vee} + q_n^{+}\delta w^{+\vee}\}A_1^{+}A_2^{+}d\alpha_1^{+}d\alpha_2^{+} +$$

$$+\iint_{S^-}\{T_1^{\vee -}\delta\varepsilon_2^{\vee -} + T_2^{\vee -}\delta\varepsilon_2^{\vee -} + S^{\vee -}\delta\omega^{-\vee} + ic_0T_2^{\vee -}\delta\varkappa_1^{-\vee} + ic_0T_1^{-}\delta\varkappa_2^{-\vee} - ic_0S^{\vee -}\delta\tau^{\vee -}\}A_1^{-} -$$

$$-A_2^{-}d\alpha_1^{-}d\alpha_2^{-} - \iint_{S^-}\{q_1^{-}\delta u_2^{\vee} + q_2^{-}\delta v^{\vee -} + q_n^{-}\delta w^{\vee -}\}A_1^{-}A_2^{-}d\alpha_1^{-}d\alpha_2^{-} = 0$$

This equation differs from equation (12) by the fact that here the integrals along the boundaries of regions S^+ and S^- ($\alpha_1^{+} = \alpha_1^{+0}$, $\alpha_1^{-} = \alpha_1^{-0}$), corresponding to the line of elastic coupling l, are missing. As above, integration by parts yields the particular Euler equations for each region. Beside that, the following integral coupling conditions will hold along the line of coupling:

$$\int_{\alpha_1^{+} = \alpha_1^{+0}}\{T_1^{\vee +}\delta u^{+\vee} + T_{12}^{\vee\prime +}\delta v^{+\vee} + N_1^{\vee\prime +}\delta w^{+\vee} + M_1^{\vee +}\delta\vartheta^{\vee - +\vee}\}A_2^{+}d\alpha_2^{+} +$$

$$+\int_{\alpha_1^{-} = \alpha_1^{-0}}\{T_1^{\vee -}\delta u^{-\vee} + T_{12}^{\vee\prime +}\delta v^{-\vee} + N_1^{\vee\prime -}\delta w^{\vee -} + M_1^{\vee -}\delta\vartheta^{\vee -}\}A_2^{-}d\alpha_2^{-} = 0 \qquad (13)$$

Let $\Delta_x^{\vee}$ and $\Delta_z^{\vee}$ be the components of a complex displacement along any two mutually perpendicular directions, lying in a plane that is perpendicular to the tangent to l (see Fig.3):

$$\delta u^{+\vee} = (e_1^+, e_x^-)\,\delta\Delta_x^\vee + (e_1^+, e_z)\delta\Delta_z^\vee$$
$$\delta w^{+\vee} = (e_n^+, e_x^-)\,\delta\Delta_x^\vee + (e_n^+, e_z)\,\delta\Delta_z^\vee$$
$$\delta u^{-\vee} = (e_1^-, e_x)\,\delta\Delta_x^\vee + (e_1^-, e_z)\delta\Delta_z^\vee$$
$$\delta w^{-\vee} = (e_n^-, e_x)\,\delta\Delta_x^\vee + (e_n^-, e_z)\delta\Delta_z^\vee$$
$$\delta v^{+\vee} = -\,\delta v^{-\vee} = \delta v^\vee$$
$$\delta\vartheta^{+\vee} = -\,\delta\vartheta^{-\vee} = \delta\vartheta^\vee$$

Substituting these relations into (13) and noting that $A_2^+ d a_2^+ = A_2^- d a_2^- = ds_2$ one obtains

$$\int_e \{(Q_x^{\vee\prime +} - Q_x^{\vee\prime -})\,\delta\Delta_x^\vee + (Q_z^{\vee\prime +} - Q_z^{\vee\prime -})\,\delta\Delta_z^\vee + (T_{12}^{\vee\prime +} - T_{12}^{\vee\prime -})\,\delta v + (M_1^{\vee +} - M_1^{\vee -})\,\delta\vartheta\}\, ds_2 = 0$$

where

$$Q_x^{\vee\prime +} = (e_1^+, e_x)\, T_1^{\vee +} + (e_n^+, e_x)\, N_1^{\vee\prime +}$$
$$Q_z^{\vee\prime +} = (e_1^+, e_z)\, T_1^{\vee +} + (e_n^+, e_z)\, N_1^{\vee\prime +}$$
$$-\,Q_x^{\vee\prime -} = (e_1^-, e_x)\, T_1^{\vee -} + (e_n^-, e_x)\, N_1^{\vee\prime -}$$
$$-\,Q_z^{\vee\prime -} = (e_1^-, e_z)\, T_1^{\vee -} + (e_n^-, e_z) N_1^{\prime -\vee}$$

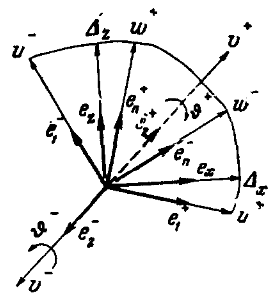

Fig. 3.

By virtue of the arbitrariness of the variations $\delta\Delta_x^\vee$, $\delta\Delta_z^\vee$, $\delta v^\vee$, $\delta\theta$ the natural complex coupling conditions are obtained:

$$Q_x^{\vee\prime +} = Q_x^{\vee\prime -}, \qquad T_{12}^{\vee\prime +} = T_{12}^{\vee\prime -}, \qquad Q_z^{\vee\prime +} = Q_z^{\vee\prime -}, \qquad M_1^{\vee +} = M_1^{\vee -} \tag{14}$$

The above statements are illustrated by the following example.

Two symmetrically loaded shells of revolution of equal thickness coupled along a general parallel circle will be investigated. In this case (see Fig. 4 and Ref. 1, p. 241), (in Figs. 3 and 4 the superscript $^\vee$ has been omitted):

$$\alpha_1^+ = \theta, \qquad \alpha_1^- = -\,\theta, \qquad \alpha_2^- = \varphi, \qquad \alpha_2^- = -\,\varphi$$
$$A_1^+ = R_1^+ = R_1^+(\theta), \qquad A_2^+ = (R_2 \sin\theta)^+ = r(\theta)$$
$$A_1^- = R_1^- = R_1^+(\theta), \qquad A_2^- = (R_2 \sin\theta)^- = r(\theta)$$

Assuming that the shell does not undergo torsion, one obtains

$$v^{\vee +} = v^{-\vee} = T_{12}^{\vee +} = T_{12}^{\vee -} = 0$$

The relations (14) in the given case are written as

$$Q_x^{\vee +} = Q_x^{\vee -}, \quad Q_z^{\vee +} = Q_z^{\vee -}, \quad M_1^{\vee +} = M_1^{\vee -} \tag{15}$$

Here, as can be seen from Fig. 4:

$$Q_x^{\vee +} = \cos\theta^+ T_1^{\vee +} + \sin\theta^+ N_1^{\vee +}, \qquad Q_x^{\vee -} = \cos\theta^- T_1^{\vee} - \sin\theta^- N_1^{\vee -}$$
$$Q_z^{\vee +} = -\sin\theta^+ T_1^{\vee +} + \cos\theta^+ N_1^{\vee +}, \qquad Q_z^{\vee -} = -\sin\theta^+ T^{\vee -} - \cos\theta^- N_1^{\vee -} \tag{16}$$

Note that here the quantity $Q_z^{\vee}$ differs in sign from the one introduced in [1]. From Fig. 4 one can also see that the real parts of $\Delta_x^{\vee}$, $\Delta_z^{\vee}$, $Q_x^{\vee}$, $Q_z^{\vee}$ in this case are the horizontal and vertical displacements and forces, respectively.

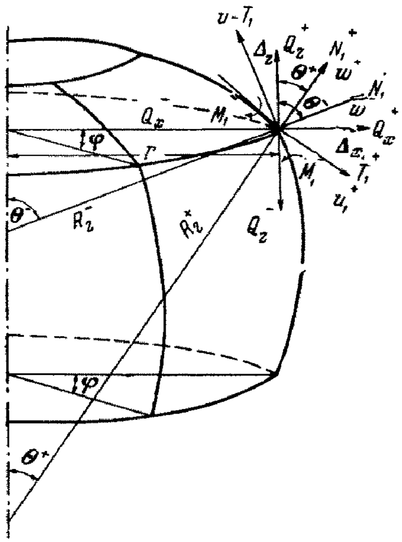

Fig. 4.

To explain the statically-geometric meaning of conditions (15) obtain from (3) and (10)

$$\varepsilon_2 = \frac{u\cos\theta + w\sin\theta}{R_2\sin\theta} = \frac{\Delta x}{r}, \quad \varkappa_2 = \frac{\cos\theta}{R_2\sin\theta}\vartheta = $$
$$= \frac{\cos\theta}{r}\vartheta, \quad \zeta_2' = \frac{\vartheta}{R^2} = \frac{\sin\theta}{r}\vartheta$$

From this

$$T_1^{\vee} = T_1 - iEhc_0\frac{\cos\theta}{r}\vartheta$$

$$N_1^{\vee} = N_1 - iEhc_0\frac{\sin\theta}{r}\vartheta, \; M_1^{\vee} = M_1 + iEhc_0\frac{\Delta x}{r}$$

Substituting the first two of these relations into (16) one obtains

$$Q_z^{\vee} = Q_z, \qquad Q_x^{\vee} = Q_x - iEhc_0\frac{\vartheta}{r}$$

$$M_1 = M_1 + iEhc_0\frac{\Delta x}{r}$$

Inasmuch as $h^+ = h^- = h$, it can be easily seen from the relations given above that the coupling conditions (14) are equivalent to the requirement for the continuity of the five real quantities Q_x, Q_z, M_1, Δ_x, θ along the line of contact.

The value Q_z usually is determined from the equilibrium condition of the shell as a whole. The second and third complex equations can be used to determine the complex constants (solutions) without having to separate them into real and imaginary parts.

The conditions $Q_x^{\vee +} = Q_x^{\vee -}$, $M_1^{\vee +} = M_1^{\vee -}$ were found previously by matching.

Everything that was said here can be generalized without any particular difficulties to the case of a Poisson's ratio μ other than zero and to shells whose middle surfaces have boundaries that do not coincide with lines of curvature.

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